ACTIVITIES FOR LINEAR REGRESSION

From the
NCSSM Statistics Leadership Institute 2000

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George Box first designed the helicopters used in this activity for teaching statistics. This specific activity can be used in the unit on inference for linear regression. Students are asked to design an experiment to collect data, fit a linear model to the data, check the assumptions for regression and inference on the slope of the regression line, perform a hypothesis test, and construct a confidence interval estimate of the slope. Alternatively, students could collect the data when studying linear regression and complete the Descriptive Statistics section, then revisit the data later when studying inference for regression. This packet is organized as follows:

- A student activity package.
- Sample data and answers (in italics) for the student activities: This section provides sample results from the experiment.
- Teacher commentary: This section provides an in-depth discussion of the issues involved in the design and analysis of this activity.
- Appendix A: This section examines data collected under an alternative experimental design.
- Appendix B: This section provides the helicopter template and directions for constructing the helicopter.
- Appendix C: Bouncing ball activity: This section provides an additional or alternative activity that will illustrate the same ideas of linear regression and inference for regression.

HELICOPTER FLIGHTS

The Question
How does the mean time it takes for a helicopter to fall to the floor change as the height from which it is dropped increases?

Materials
- Helicopter Template
- Stopwatches
- Measuring tapes or meter sticks
- Paper clips (for varying weight) or medium-sized binder clips for outdoor drops
- Masking tape (to mark drop heights on vertical wall)
- Cards, dice, or random digit table

Experimental Design
Describe in detail how you will collect your data. Include the number of trials you will conduct at each height, the heights you will use, and how you will randomize these. Also address the issue of how the helicopters will be dropped (technique for dropping and who will do the drop) and how the descent will be timed (number of timers). Explain possible sources of variability in your data and efforts you will make to reduce effects of possible confounding variables. Describe the scope of inference for your results.

Data Collection
As your group collects data, record the drop height and time for each drop. You may use the attached Student Data Sheet or create a similar data sheet to be consistent with the design of your experiment.

STUDENT DATA SHEET

(Height is measured in meters, time in seconds)

Trial	Height: ____	Height: ____	Height: ____	Height: ____	Height: ____
1					
2					
3					
4					
5					
6					
7					
8					
9					
10					
11					
12					
13					
14					
15					
16					
17					
18					
19					
20					

Descriptive Statistics

1. Construct a scatterplot of the response variable (time of descent) versus the explanatory variable (height) and copy it below. Does a linear model appear to be appropriate for these data? Explain.

2. Find a least squares regression line. Graph the line on the scatterplot and make a scatter plot of the residuals. Do you feel that a linear model is appropriate? Explain using the residual plot. What does the slope tell you about the relationship between drop height and descent time? What is the meaning of the y-intercept?

Inferential Statistics

1. Checking the assumptions: For each drop height, construct a histogram to display the distribution of times of descent. Calculate and record the sample mean and standard deviation of these distributions. Are the assumptions of normality and equal variance across values of the explanatory variable valid with these data? Explain. Also discuss any other assumptions necessary for inference on regression. Be sure to explain why you think each assumption is or is not satisfied.

2. Provide a copy of your computer output. Specifically include the parameter estimates and the standard errors of the estimates.

3. Perform a hypothesis test for the slope of the regression line. Be sure to state your hypotheses in both symbols and words.

4. Construct a confidence interval estimate of the slope β. Show all work.

5. Conclusions. State your conclusions in complete sentences, summarizing the results of the hypothesis test and confidence interval in the context of the question you are trying to answer.

HELICOPTER FLIGHTS
SAMPLE STUDENT ANSWERS

The Question: How does the mean time it takes for a helicopter to fall to the floor change as the height from which it is dropped increases?

Note to teacher: The following provides results from when we ran this experiment at the NCSSM Statistics Institute. The specific experimental design that we used and the results we obtained are provided in this section. For a discussion of alternative designs and consequent changes in the conclusions we can draw, see the Teacher Commentary. The design and results were written from a student's point of view.

Experimental Design

Describe in detail how you will collect your data. Include the number of trials you will conduct at each height, the heights you will use, and how you will randomize these. Also address the issue of how the helicopters will be dropped (technique for dropping and who will do the drop) and how the descent will be timed (number of timers). Explain possible sources of variability in your data and efforts you will make to reduce effects of possible confounding variables. Describe the scope of inference for your results.

Data Collection:

To examine the relationship between the drop height and descent time, we dropped helicopters from each of five different heights and measured the descent time.

- *We constructed 50 long-rotor helicopters. (See template for constructing helicopters in Appendix B.) We used copy paper and placed a jumbo paper clip on the shaft to provide more stability.*
- *We selected and marked five drop heights in a gym: 1.77 m, 2.29 m, 2.83 m, 3.36 m 4.39 m.*
- *To organize the data collection, we worked in pairs. Ten people collected data, giving 5 pairs. In each pair, one acted as a dropper and the other as a timer and then the roles were reversed.*
- *Each of us dropped one helicopter from each height, using a different helicopter for each drop. Because we were working in pairs, each timer also measured at each height. This resulted in 10 trials at each height.*
- *Each pair randomized the order of height and dropper, using a calculator random number generator.*
- *The dropper held the helicopter under the rotors with the base even with the height marker.*
- *The timer measured the time it takes from when the helicopter is released until it hits the ground. Times were recorded to the nearest hundredth second.*
- *We combined the data from all students for the analysis.*

Sources of Variability and Efforts to Reduce Effects of Confounding Variables:

- *Differences in droppers and timers: Because everyone drops and times at each of the heights these differences should not be confounded with height though they will lead to additional variation in times reported.*
- *Differences in helicopters: Because the helicopters were constructed in the same manner and were selected randomly, helicopter condition should not be confounded with height if a new helicopter is used for each drop. The use of different helicopters may also increase variation in times reported. However, repeated use of helicopters could cause problems due to wear*

and tear that we felt were more difficult to control and repeated use restricts the scope of inference.

- *Weather conditions (temperature, humidity, and altitude): Attempts should be made to control these conditions by performing the experiment in one location and gathering all data at the same time. We performed the experiment inside because conditions were easier to control than if we had performed the experiment outside, where breezes could be a confounding variable.*

Scope of Inference:

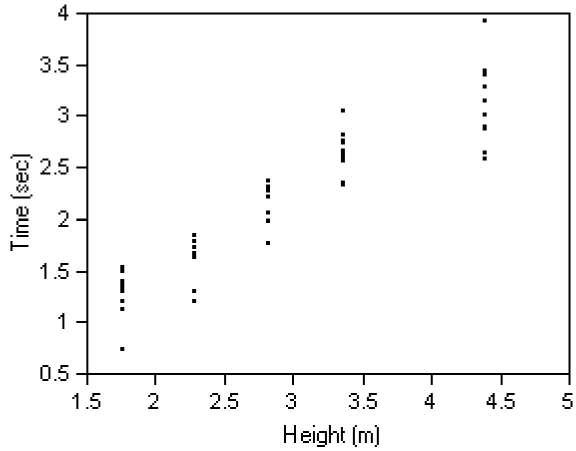
- *Since all helicopters were constructed from the paper in one package of paper, it is only possible to make inferences about helicopters constructed from that package of paper. However, it seems reasonable to expect helicopters made from copy paper of the same weight to behave in a similar manner, so we feel comfortable extending our scope of inference to include other packages. It would not be reasonable to make inferences about helicopters constructed from paper that was either lighter or heavier based on our data.*
- *The time of descent may be predicted from the height of the drop for heights ranging from approximately 1.77 m to 3.36 m. It would be risky to try to predict for heights either above or below those values.*
- *The drops were performed during the month of July in Durham, NC. It is reasonable to believe that locations with different altitudes, temperature, or humidity levels may produce different results.*

Actual Sample Data (Height is measured in meters, time in seconds)

Trial	Height: 1.77	Height: 2.29	Height: 2.83	Height: 3.36	Height: 4.39
1	1.49	1.77	2.36	2.60	3.38
2	1.12	1.62	2.04	2.65	3.42
3	1.39	1.66	2.27	2.80	3.91
4	1.19	1.64	2.30	2.58	3.13
5	1.35	1.28	1.75	2.31	2.63
6	1.32	1.20	1.97	2.33	2.58
7	1.28	1.63	2.26	2.73	2.86
8	0.73	1.83	2.20	2.74	2.89
9	1.32	1.72	2.27	2.56	3.27
10	1.53	1.64	2.29	3.04	2.99

Descriptive Statistics

1. Construct a scatterplot of the response variable (time of descent) versus the explanatory variable (height) and copy it below. Does a linear model appear to be appropriate for these data? Explain
   Answer:



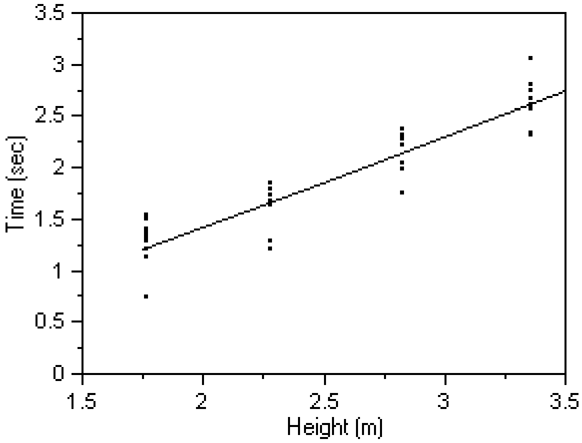
Although there is variation in the response variable at each height, there appears to be a positive linear relationship between height and mean time of descent. The points at the greatest height appeared to be more spread than the lower heights. Because of these differences, we interviewed the people involved in collecting the data, and determined that there were different conditions affecting the drops from that height. Specifically, individuals positioned their helicopters much differently because the height was not labeled, some students dropped at one end of the gym and others at the other end. Furthermore, this drop was from the balcony, not from the bleachers like all of the other heights. Therefore, we decided to omit the data collected at this height for further analysis. Thus our scope of inference is narrowed, making it risky to predict times for heights exceeding 3.36 meters.

Note to teacher: It is important to note that you should be very careful when making a decision to omit data. Getting a better fit for your model is not a legitimate reason to exclude information!

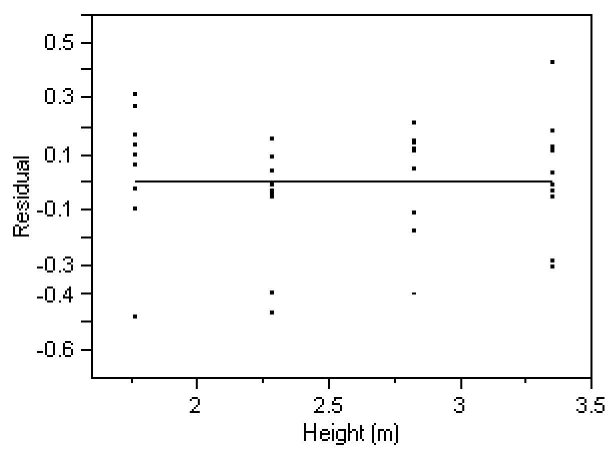
2. Develop a linear model for the data. Graph the line on the scatterplot and make a scatter plot of the residuals. Do you feel that a linear model is appropriate? Explain using the residual plot. What does the slope tell you about the relationship between drop height and descent time? What is the meaning of the y-intercept?

Since the data appear to be linear, a least squares regression line was fit to the data.

SCATTERPLOT:



PLOT OF RESIDUALS:



Time = -0.330329 + 0.877787 Height

The plot of residuals does not suggest any major problems with the linear model. At each height, both positive and negative values may be observed and there is no obvious pattern to the residuals. The envelope, or outermost points, in the residual plot appears curved; however, if one covers the residual point in the upper right corner, the apparent curvature disappears. This suggests that there is no true pattern of curvature -- only an envelope that suggests curvature to the inexperienced eye.

The slope of the regression line tells you that for every one-meter increase in drop height, the descent time increases by approximately .88 sec on average. The reciprocal of the slope is an estimator for the velocity of the helicopters. This velocity is therefore estimated at 1.13923 m/sec on average. The y-intercept would be the time of descent when the initial height is zero. This value certainly should be zero. Since it is not zero, this is an example of the dangers of extrapolating outside the range of the data. For further discussion, see the teacher commentary.

Note to teacher: If you only look at the largest residual at each height, there appears to be curvature. The eye is drawn to the outer most points in any plot. However, looking at all points, a linear model seems appropriate because the positive and negative residuals were balanced at each height. For further discussion, see the teacher commentary.

Inferential Statistics

1. Checking the assumptions: Construct a histogram of the distributions of observed times of descent for each height. Calculate and record the sample means and standard deviations for each height. Are the assumptions of normality and equal variance appropriate with these data? Explain. Also discuss any other assumptions necessary for inference on regression.

   (See the last page of this section for graphs. Sample means and standard deviations are provided with the graphs.)

   Normality: For each height, a histogram, boxplot, and normal quantile plot has been constructed. The lowest height (1.77 m) has a low outlier, otherwise the distribution appeared to be fairly normal. The distributions at 2.29 m and 2.83 m are suspect, but only one point falls outside the confidence band for the normal quantile plots. The measurements at 3.36 m and 4.39 m look fairly normal. The small sample size made assumptions for normality somewhat difficult to judge, but no serious concerns seem to be present.

   Constant standard deviation of the response variable across values of the explanatory variable: The difference in the sample standard deviations for the four heights included in the analysis was not large enough to be a concern.

   Data were collected using techniques consistent with a SRS. The helicopters were constructed from paper taken from one package of copy paper, then selected randomly before they were dropped. It is reasonable to believe there was independence between the drops.

   A linear model seems appropriate for these data as discussed earlier.

   Note to Teacher: It should be noted that normality is not a requirement for descriptive statistics. It is needed when conducting tests and calculating confidence intervals for slope. Also, computer packages usually use the terms mean, variance, and standard deviation without regard to whether population parameters or their sample estimates are being computed. It is the responsibility of the experimenter to communicate clearly whether population or sample quantities are being considered.

2. Parameter Estimates (from computer output or calculator):

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	-0.330329	0.146986	-2.25	0.0305
Height	0.877787	0.05588	15.71	<.0001

3. Perform a hypothesis test for the slope of the regression line. Be sure to state your hypotheses in both symbols and words.

   H_0: $\beta = 0$ *(On average the descent time of the helicopters is the same regardless of the height from which they are dropped.)*

   H_a: $\beta > 0$ *(On average the descent time of the helicopters increases as drop height increases.)*

   $$t = \frac{b}{se_b} = \frac{0.877787}{0.05588} = 15.71$$ *(b and se_b come from the computer output)*

$p < \frac{.0001}{2} = .00005$ *(The computer reports the results for a two-sided test. Since this is a one-sided test, the p-value from the compute output was divided by 2.)*

4. Construct a confidence interval estimate of β. Show all work.

   95% Confidence Interval: $b \pm t^* (se_b) = 0.877787 \pm 2.025 * 0.05588 = (.764792, .990782)$

   The t-value is based on 40 – 2 = 38 degrees of freedom. We are 95% confident that the mean change in time per unit change in height is between 0.765 sec/m and 0.991 sec/m. A confidence interval for the velocity was calculated by taking the reciprocal of the endpoints of the confidence interval for the slope. With 95% confidence, the mean velocity is between 1.009 m/sec and 1.307 m/sec.*

5. Conclusions: State your conclusions in complete sentences, summarizing the results of the hypotheses test and confidence interval in the context of the question you are trying to answer.

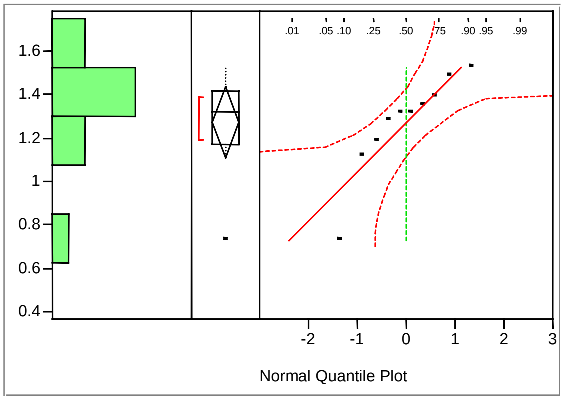
   Based on the small p-value, we can conclude that it is highly unlikely we would get this large or larger positive slope from a regression on a random sample if the true value of β were zero. The confidence interval contains only positive values, supporting the conclusion drawn by our hypothesis test that the true slope β *is greater than zero.*

   Therefore, we can conclude that there is a positive relationship between time and height, i.e. greater heights produce larger descent times. Because of the linear nature of the data, descent time can be predicted from height of drop using the least squares regression line.

DISTRIBUTION OF TIME BY HEIGHT

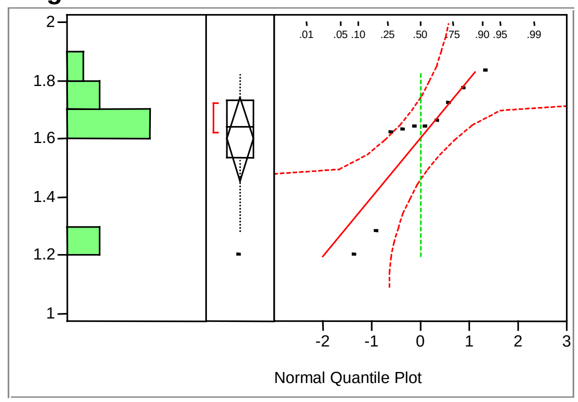
The following graphs were made using JMP IN (pronounced "jump in"). They were provided to give the teacher additional insight into the analysis. It is not necessary that you have JMP IN to perform this experiment. Other statistical packages and some graphing calculators will give similar plots

Height: 1.77m



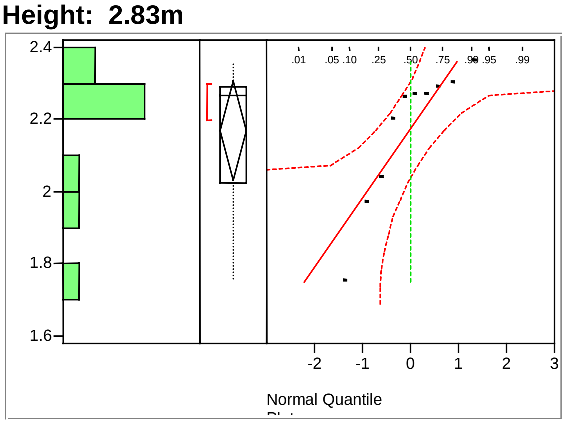
Mean 1.272000
Std Dev 0.226657

Height: 2.29m



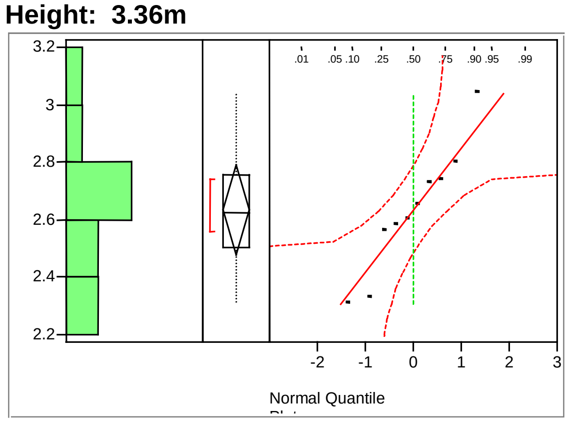
Mean 1.599000
Std Dev 0.201850

Height: 2.83m



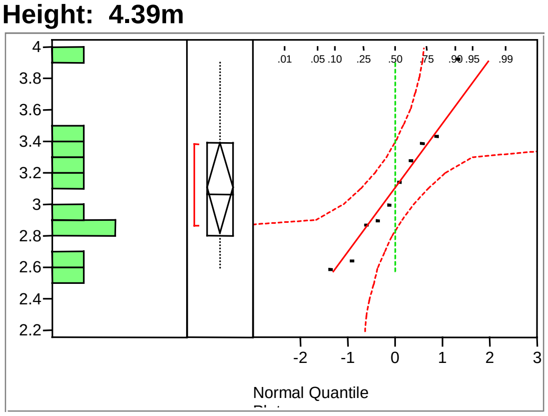
Mean 2.171000
Std Dev 0.191395

Height: 3.36m



Mean 2.634000
Std Dev 0.216035

Height: 4.39m



Mean 3.106000
Std Dev 0.404343

Teacher Commentary

Questions about the Logistics of the Experiment

- *Will this experiment really produce linear data?* If we examine the relationship between descent time and height, we might initially think the relationship will be based on the physical relationship $h = \frac{1}{2}gt^2 + h_0$, where g is the acceleration due to gravity. Solving for time, $t = \sqrt{\frac{2(h - h_0)}{g}}$. This equation is not linear -- but it is based on the assumption of no air resistance. Clearly air resistance will affect the time it takes the helicopter to drop. The helicopter will briefly "flutter", with its velocity changing; then when it begins whirling, it very quickly reaches a speed that remains constant, its terminal velocity. Therefore, in practice, helicopters should be dropped from at least 1.5 to 2 meters to allow sufficient time for them to reach terminal velocity. The "fluttering time" should be (on average) constant for all drops from all heights. Thus, differences in descent times will be due to different heights, and because the terminal velocity is constant the relationship between height and falling time should be linear.

  This "fluttering" also has an effect on the *y*-intercept. It might seem that the *y*-intercept represents the time it would take a helicopter to fall from zero height, which "should" be zero. The fact that it is not zero (and is likely to be significantly non-zero for real data) is evidence of the danger of extrapolation beyond the domain of the data.

  The cause of the non-zero intercept is the short period of time that the helicopter falls before reaching a constant terminal velocity. The helicopter generally falls fast for a short time before it begins spinning, and then slows to its terminal velocity. Thus, the total flight time is likely to be less than it would have been had it traveled at a constant terminal velocity for the entire descent. That is why the model of descent time as a function of height is likely to have a negative constant term.

- *What type of paper should we use?* In this experiment, we used regular copy paper, stapled the base together, and attached a jumbo paper clip to the base of the helicopter. The paper clip provided the extra weight necessary to help the helicopter reach a steady state. Copy paper helicopters without any weights will not rotate as consistently as those with some weight attached. You may also want to use card stock paper or construction paper so the helicopters last longer. The type of paper you use may affect your descent times.

- *From what heights should the helicopters be dropped?* In this experiment, we used different levels of the bleachers in the gym to determine the heights. The heights were separated by two rows of bleachers. In addition, the balcony was used for the highest drop height. However, after collecting the data, we realized the conditions were not quite the same for the highest drop. The heights were selected because they were easy to use. However, the lowest height should be selected carefully. If it is too low, the helicopter may not reach terminal velocity, and therefore the time it takes to reach the ground may not follow the linear trend of the other heights, as discussed above.

- *Would it be appropriate to have three timers for each height and take a mean or median measure for these times?* Many students are familiar with this technique from science classes or sporting events. It is appropriate to use this technique and it will probably reduce the variability of times because some of the measurement error will be removed. The regression model is for the average true descent time. What we can measure is "true descent time + stop watch error, etc." If you know the "stop watch error" is not symmetric, the median of the three measurements should be used.

- *How long did the experiment take?* If you only have 45-minute periods, we suggest that you discuss the experimental design issues and perform some practice runs on the first day so that the design, the heights, and the dropping and timing techniques can be finalized before starting the full experiment. The second day would be the actual data collection, which could take the entire period. At the end of this period, students could be given the class data to take home and start the preliminary analysis or write up a full analysis outside of class.

Questions about the Experimental Design

The challenging part of teaching experimental design is that students can come up with a wide variety of designs. As the teacher, you must determine not only if they have good designs, but also whether the data collected from these designs will answer the question posed and what type of analysis should be performed on the data. The following discussion examines some changes in the experimental design and how these changes would affect the conclusions we can draw and the analysis we would perform.

- *What if only one pair of students collects the data?* If one student drops all helicopters and another does all the timing, variability would be reduced. However, the scope of inference is limited to drops by those two students. The time required to collect the data may be prohibitive for a class period because the pair will be collecting all the measurements and must move from height to height in random order. In addition, to keep students engaged, it is better to have everyone collect some data. To study this design in more detail, we collected data using one dropper and one timer. A discussion of the results is provided in the Appendix A.

- *What if you stationed one pair of students at each height and had the pair collect all of the data at that height?* This method would certainly reduce the time it takes to collect the data. However, it creates a major issue in the analysis. Differences in timing and dropping between the pairs will likely create confounding variables that might not allow us to see a linear relationship. In addition, a major assumption in inference on slope is that the times at each height will be normally distributed with the same variance. Although we did not test this method, it is possible that different pairs will have different variability and therefore, the assumptions underlying the inference techniques may not be met.

- *What if you didn't require each group to drop from each height, rather the pairs randomly selected 10 heights and dropped the helicopters from the heights in the order selected?* In this scheme it is possible that each student may not drop from every height. However, this design is valid and similar techniques to those provided in the Sample Student Answers can be used.

Questions about the Analysis - Descriptive Statistics

- *When is it appropriate to remove data from the analysis?* As discussed in the Sample Answers, we must be very cautious about removing data. In this experiment, when we first examined the data, we could see that times for the four lowest heights behaved much more consistently than times for the highest drop height. Thus, we reviewed our data collection at this height and found that individuals dropped at different positions on the balcony and at slightly different heights. In addition, all of the other heights were collected from the same part of the gym. There were enough differences in the data collection at the highest height that we felt we should limit our scope of inference for heights to the four lowest heights. We emphasize that removing data just because it does not fit the linear trend is not appropriate.

- *What are reasonable guidelines for examining residuals and patterns?* Residuals provide the major tool in assessing the linearity of the data. Students should look for patterns in the residuals that imply a linear relationship is not appropriate. An obvious pattern that implies a linear function is not appropriate is residuals that follow the trend of a parabola. Occasionally, it is difficult to determine whether or not a pattern is present—what one student recognizes as a pattern, another student might suggest is random. If you consider your *x*-values in three groups (low, medium, and high), you can use the following guideline: check to see that the residuals are relatively balanced between positive and negative values for each of the three groups

- *Can we use the line to make predictions?* Because we are comfortable that the mean descent times are linear for heights above that needed for the initial flutter, we can use the line to predict the mean descent time for different heights within the range of heights we used (interpolation). However, we should not use the line to predict the descent times for heights that are outside of the range of heights (extrapolation).

Questions about the Analysis – Inferential Statistics

- *What do the assumptions mean and what are reasonable guidelines for accepting the assumptions?* A major focus of this experiment is to illustrate the requirements for valid inferential statistics on regression:

  - <u>There is a linear relationship between the *x*-values and the means of the *y*-values.</u> In this experiment, we examined whether the data were linear by considering the physical reality we were exploring as discussed earlier in the teacher commentary. The data supported linearity as well. Typically scientists will have a strong theory on the type of function that should be used to model the data. When possible, theoretical models should be discussed with students before moving quickly to analysis of the data. As discussed above, residuals should be examined thoroughly to determine whether a linear model is appropriate.

  - <u>At each *x*-value there is a population of *y*-values that is normally distributed.</u> This experiment allows students to examine the normality of the data at each *x*-value because there are multiple trials at the same height. With only 10 values at each *x*-value, a smooth bell-shape is unlikely. However, students can examine a histogram or boxplot of the data, look for outliers and extreme skewness in the data. Because the t-procedures are robust, some departure from normality is acceptable. More measurements at each height will enable you to determine more easily whether this assumption is met. Frequently, in practice, the data do not include multiple *y*-values at each *x*-value. If this is the case, we can examine the normality of the residuals. If the residuals are relatively normal, we feel more comfortable with the assumption that the y-values are normally distributed at each x-value because in a random selection of *y*-values, we should see the same variation around the line as we should see for each *x*-value.

- <u>The variance in *y*-values at a given *x*-value is constant across all *x*-values.</u> The scatterplot produced in this experiment provides a nice visual to understand this concept. We can see in the graph the range in y-values for each *x*. With the exception of the fifth height, the range in *y*-values is very consistent. Based on the sample size used in this experiment, the largest variance should be no more than about four times the size of the smallest variance.

- <u>The sample *y*-values are independent.</u> This assumption connects to the idea that there is a population of *y*-values for each *x*. We are assuming that individual observations from this population of *y*-values are independent of each other. The values are centered around the population mean and the population is normally distributed; however, the difference between an individual y and the mean *y*-value—the error or residual—is a random variable. Thus the *y*-values are independent.

- *Why didn't we discuss all of these assumptions when we studied regression as a Descriptive Statistics topic?* Inference on regression is based on the same ideas as inference on means or proportions: we are using sample data to draw conclusions about or make estimates of the population parameters. In linear regression, we are estimating two parameters—the slope and the *y*-intercept—to estimate the population regression line. It is important to consider that there are two lines—the "true" population line and the line estimated using the sample data. The population regression line models the change in the <u>mean</u> y-value for corresponding changes in *x*. It is not intended to model all *y*-values at each *x*-value. If we are not intending to perform a statistical test on or build a confidence interval for the slope and *y*-intercept, these assumptions are not important: we can use the line to describe the trend and to make predictions within the range of our *x*-values.

- *How should my students interpret the results of a t-test for the slope of the regression?* This *t*-test is used to assess whether or not the "true" slope is zero. It does not tell us whether or not the relationship is linear; the residual graph, as discussed above, is our primary tool in assessing linearity. However, students should go beyond stating that the slope is zero or the slope is not zero. In this experiment, we were examining if the descent time increases as the height increases. Because we showed that the slope was significantly positive, we concluded that the descent time does increase as descent height increases and we can estimate, on average, how much the descent time will change with changes in drop height. If we failed to reject the null hypothesis, we would conclude that there was not sufficient evidence to suggest that the descent time changed with changes in height.

- *Why don't we test the significance of the y-intercept?* In our experiment, the computer output included a *p*-value of 0.0305 for the *y*-intercept, implying that it was statistically different from zero. However, zero is well below the range of heights we studied and, as discussed earlier, we should not drop the helicopters from heights that are too low because the linear relationship may not hold. Furthermore, the *y*-intercept would represent the time it would take the helicopter to fall if it were dropped from a height of zero. Therefore, the intercept is not very valuable in this situation and hence, we did not use inferential techniques on the intercept. In many other situations, it is appropriate to examine the significance of the *y*-

intercept. For example, if we were modeling salary based on years of experience, we could estimate the mean starting salary using the *y*-intercept.

- *How should my students interpret the results of the confidence interval?* The confidence interval provides an estimate for the slope parameter. It is important to reiterate that this estimate is based on sample measurements.

Appendix A - What if you only used one dropper?

In our suggested design, we used several droppers and required that each dropper drop helicopters from each height. Another design might have only one dropper and timer pair for the entire experiment. This dropper would drop multiple times from each height. It is still very important that the dropper randomize the heights and not just drop all of the helicopters from height 1, then height 2, and so on. Rather, random numbers between 1 and 5 to represent heights should be selected until each height is selected 10 times. This string of numbers will represent the order in which the heights are completed. For instance, the string 134253312 would suggest that you drop at height 1 first, then height 3, then height 4, etc.

This design has a narrowed scope of inference. Because we are using only one dropper/timer pair, the true scope of inference is only for this pair. In contrast, if several droppers are used, the scope of inference is widened to more droppers and if these droppers are random selections of students at the school, then we can draw inferences for the entire population of students at that school. We have provided the following discussion to show what differences in the data we found when a single dropper was used (Single Dropper/Timer Design) and how it compares to the data collected in our suggested design (Multiple Dropper/Timer Design).

Actual Sample Data *Single Dropper/Timer Design* – (Height is measured in meters, times in seconds)

Trial	Height 1.77 m	Height 2.29 m	Height 2.83 m	Height 3.36 m
1	1.21	1.60	1.95	2.65
2	1.17	1.30	2.17	2.39
3	1.36	1.40	2.27	2.25
4	1.30	1.48	1.64	2.33
5	1.21	1.63	1.92	2.16
6	1.20	1.49	1.76	2.46
7	1.05	1.58	1.93	2.42
8	1.25	1.34	1.91	2.42
9	1.35	1.48	1.90	2.39
10	1.25	1.39	2.17	2.43
Mean	1.235	1.469	1.962	2.390
St. Dev.	0.091	0.112	0.193	0.130
Sample Size	10	10	10	10

Multiple Dropper/Timer Design (The raw data are in the Main Report)

Mean	1.272	1.599	2.171	2.634
St. Dev.	0.227	0.202	0.191	0.216
Sample Size	10	10	10	10

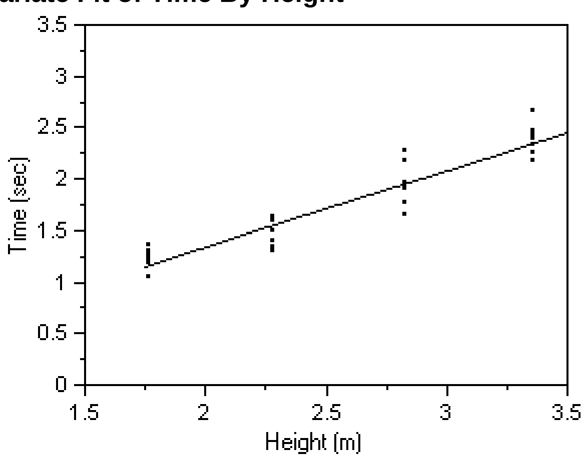
An initial review of the summary statistics illustrates two key differences in the results under these designs:

- Mean Time: The data collected under the single dropper/timer design had a lower mean time for each height than from the multiple dropper/timer design. This result could be due to differences in the timer and dropper combination. For example, an individual timer may consistently start timing late or may have better reaction times and thus, stop the timer more quickly than the other timers. Therefore, if this timer is used to collect all of the data, mean times will be consistently lower than when multiple timers are used. If a different person timed all of the drops, we easily could have had all higher mean times or had times that varied randomly around the means.

- Variation in Time: The data collected under the single dropper/timer design had much less variation in descent times than the times for the multiple droppers. With the exception of the 2.83 m height, the standard deviations are approximately twice as large under the multiple dropper/timer design. Using one dropper provides more consistency in the method used to drop. In addition, the dropper/timer pair collected much more data, and thus, had more practice in working together. Both of these factors are likely to reduce the variation in descent times.

In both designs, the population of y-values at each x-value consists of all possible times for the helicopter to drop from height x under the conditions of the study. In the original design, variability in times at a fixed height was due to differences in droppers, differences in timers, and unexplained variability. In this design, variability in times at a fixed height no longer includes differences in droppers or timers. The scope of inference is restricted to that timer and that dropper. With this restriction, a reduction in the variability in the y's (times) at a given x (heights) is gained.

Single Dropper/Timer Design
Bivariate Fit of Time By Height



Linear Fit

Linear Fit

Time = -0.147722 + 0.7460378 Height

Summary of Fit

Rsquare	0.906791
RSquare Adj	0.904339
Root Mean Square Error	0.145691
Mean of Response	1.764
Observations (or Sum Wgts)	40

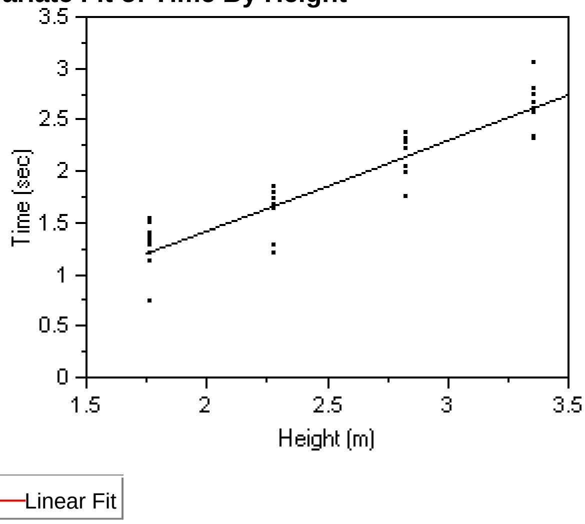
Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	1	7.8469745	7.84697	369.6880
Error	38	0.8065855	0.02123	Prob > F
C. Total	39	8.6535600		<.0001

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	-0.147722	0.102061	-1.45	0.1560
Height	0.7460378	0.038801	19.23	<.0001

Multiple Dropper/Timer Design
Bivariate Fit of Time By Height



Linear Fit

Time = -0.330329 + 0.877787 Height

Summary of Fit

RSquare	0.866552
RSquare Adj	0.86304
Root Mean Square Error	0.20982
Mean of Response	1.919
Observations (or Sum Wgts)	40

Analysis of Variance

Source	DF	Sum of Squares	Mean Square	F Ratio
Model	1	10.863229	10.8632	246.7542
Error	38	1.672931	0.0440	Prob > F
C. Total	39	12.536160		<.0001

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	-0.330329	0.146986	-2.25	0.0305
Height	0.877787	0.05588	15.71	<.0001

The scatterplots also illustrate that the mean drop time is lower under this single dropper/timer pair than the multiple dropper/timer pairs. An interesting question is whether the two lines are significantly different. The statistical techniques to answer this question are beyond the scope of the AP curriculum. However, we have included a brief discussion of the techniques and the results below. There are two questions we can answer about the differences in the lines:

1) Are the slopes significantly different?
2) Are the y-intercepts significantly different?

To answer these questions, we performed a multivariable regression on the combined data that includes an intercept dummy for the type of design used, and a slope dummy for the interaction between slope and the type of design used. A dummy variable incorporates categorical data in an equation by taking the value one for a single category and zero for the other category. Our specified model is provided below:

$$\text{Time} = \beta_0 + \beta_1\text{*DropperType} + \beta_2\text{*Height} + \beta_3\text{*DropperType*Height} + \text{error}$$

Where DropperType = 0 for the single dropper
DropperType = 1 for the multiple droppers

The variable DropperType has the effect of adding a constant to the slope and y-intercept when modeling the multiple dropper/timer design. Effectively, the above model provides two equations:

Single Dropper/Timer Design: $\text{Time} = \beta_0 + \beta_2\text{*height} + \text{error}$
Multiple Dropper/Timer Design: $\text{Time} = (\beta_0 + \beta_1) + (\beta_2 + \beta_3)\text{*height} + \text{error}$

If we find that the parameter β_1 is significantly different from zero, we can conclude that the single dropper/timer design produced a statistically different *y*-intercept than the multiple dropper/timer design. Similarly, if we find that the parameter β_3 is significantly different than zero, we can conclude that the slopes or the relationship with height is significantly different. The following output summarizes the fit of the multivariable model.

Parameter Estimates

Term	Estimate	Std Error	t Ratio	Prob>|t|
Intercept	-0.239026	0.089472	-2.67	0.0092
Height	0.8119124	0.034015	23.87	<.0001
DropperType[Class]	0.0775	0.020194	3.84	0.0003
(Height-2.5625)*DropperType[Class]	0.0658746	0.034015	1.94	0.0565

Effect Tests

Source	Nparm	DF	Sum of Squares	F Ratio	Prob > F
Height	1	1	18.587841	569.7385	<.0001
DropperType	1	1	0.480500	14.7279	0.0003
Height*DropperType	1	1	0.122362	3.7505	0.0565

The overall equation includes all of the variables:

$$\text{Time} = -0.239 + 0.0775 \cdot \text{DropperType} + 0.812 \cdot \text{Height} + 0.0659 \cdot \text{DropperType} \cdot \text{Height}$$

Single Dropper/Timer Design: Time = -0.239+ 0.812*Height

Multiple Dropper/Timer Design: Time = (-0.239+0.0775) + (0.812+0.0659)*Height
= -0.162 + 0.878*Height

The *p*-value (0.0003) associated with the variable DropperType suggests that there is a significant difference in the y-intercepts for the different designs. This conclusion is supported by the discussion above comparing the mean times at each of the heights. The mean descent times collected in multiple dropper/timer design were consistently higher; therefore, the coefficient for the variable DropperType is positive, shifting the line up.

The *p*-value (0.0565) associated with the variable DropperType*Height suggests that at the 5% significance level the slopes are not significantly different. This result suggests that the terminal velocity of the helicopters are not significantly different for the two designs.

Appendix B - Constructing the Helicopter

The following instructions use the George Box helicopter template that is included.

1. Cut out the rectangular outline of the helicopter.

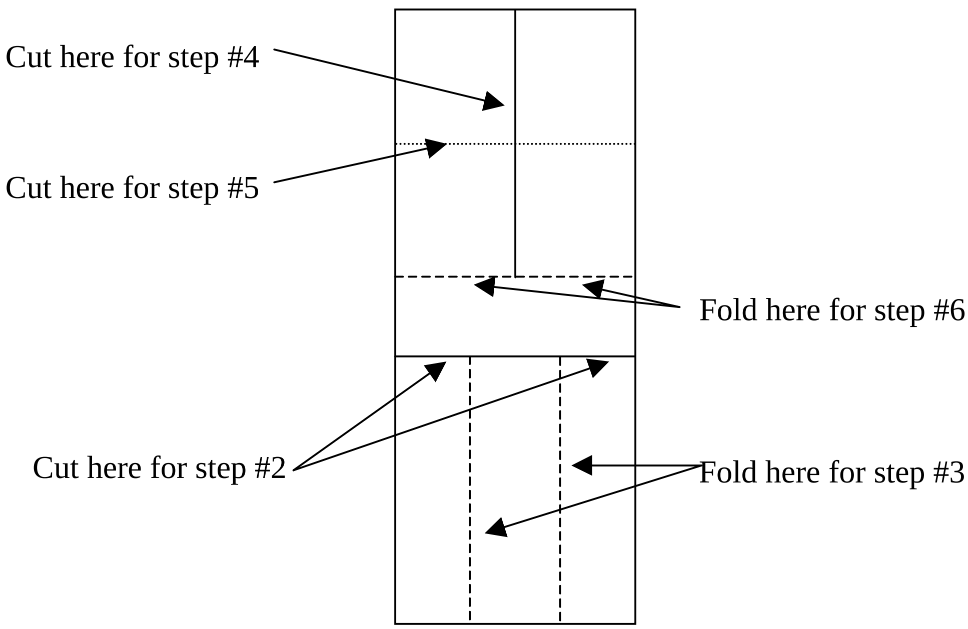
2. At the 10 cm mark, cut one-third of the way in from each side of the helicopter to the vertical dashed lines.

3. Fold both sides toward the center creating the base. The base can be stapled at the top and bottom. Try to be consistent about where the staples are placed and how many staples are used.

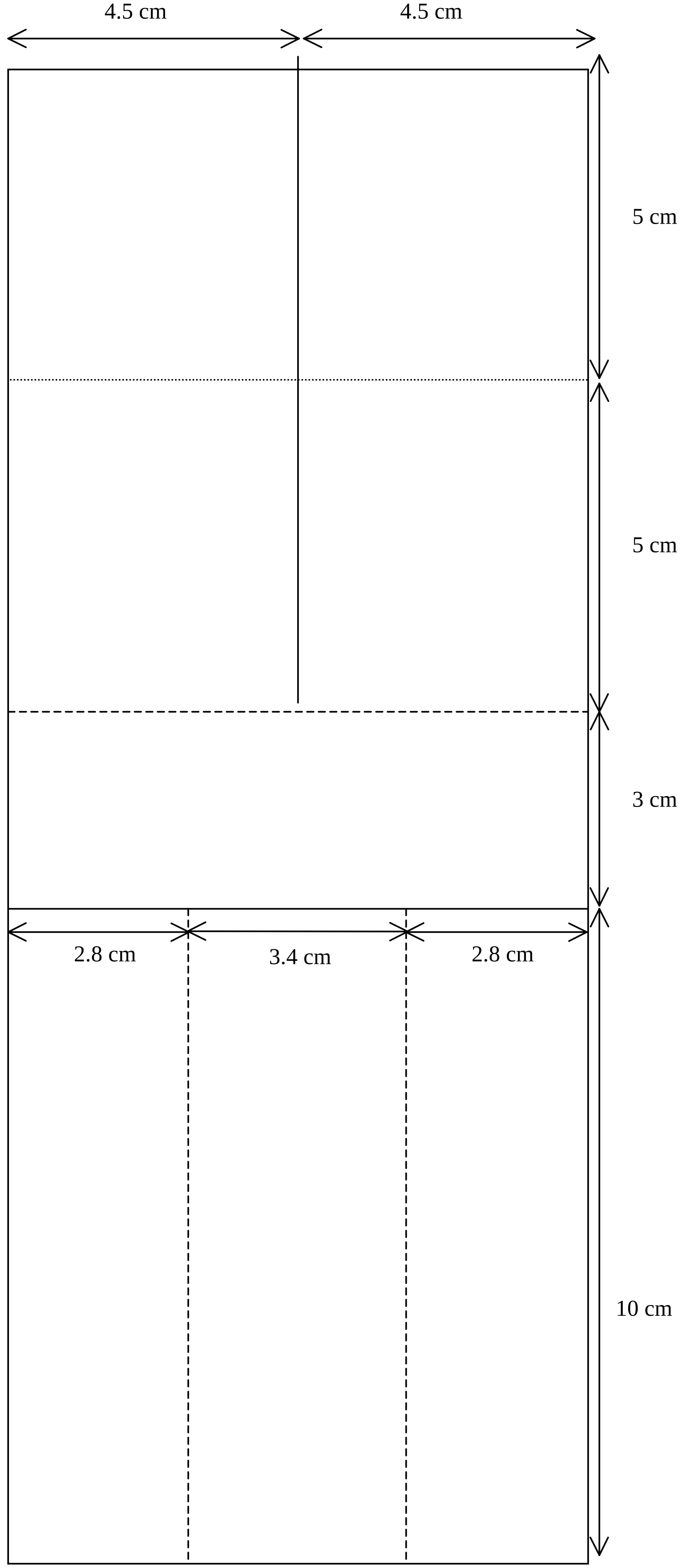
4. For long-rotor helicopters, cut down from the top along the center line to the horizontal dashed line.

5. For short-rotor helicopters, proceed as in step 4, but cut the rotors off along the horizontal line marked at 5 cm.

6. Fold the rotors in opposite directions.



PAPER HELICOPTER DESIGN
Original Design by George Box



WHAT GOES DOWN MUST COME UP
(ONLY NOT QUITE AS FAR)

QUESTION: **How does the height from which a ball is dropped affect the height of the first bounce? Can we use the initial height to predict the bounce height?**

MATERIALS:
- One ball for each group
- Stopwatches
- Tape measures taped to the wall to measure height

EXPERIMENTAL DESIGN:
A ball will be dropped repeatedly from different heights and the height of the **first** bounce will be measured. Describe in detail the design of your experiment and how you will randomize your trials. Explain possible sources of variability in your data and efforts you will make to reduce effects of possible confounding variables. Describe the scope of inference for your results.

DATA FOR BALL BOUNCE

TRIAL	HEIGHT:	HEIGHT:	HEIGHT:	HEIGHT:
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				
14				
15				
16				
17				
18				
19				
20				

Descriptive Statistics

1. Construct a scatterplot of the response variable (rebound height), versus the explanatory variable (initial height). Does a linear model appear to be appropriate for these data? Explain.

2. Find the least squares regression line for the data. Graph the line on the scatterplot and make a scatter plot of the residuals. Do you feel that a linear model is appropriate? Explain using the residual plot. What does the slope tell you about the relationship between the initial height and the rebound height? What is the meaning of the *y*-intercept?

Inferential Statistics:

1. For each drop height, construct a histogram to display the distribution of rebound heights. Calculate and record the mean and standard deviation of these distributions. Are the assumptions of normality and equal variance across values of the explanatory variable valid with these data? Explain. Also discuss any other assumptions necessary for inference on regression. Be sure to explain why you think each assumption is or is not satisfied.

2. Provide a copy of your computer output. Specifically include the parameter estimates and the standard errors of the estimates.

3. Perform a hypothesis test for the slope of the population regression line. Be sure to state your hypotheses in both symbols and words.

4. Construct and interpret a confidence interval estimate of the slope β. Show all work.

5. Conclusions: State your conclusions in complete sentences, summarizing the results of the hypothesis test and confidence interval in the context of the question you are trying to answer.

TEACHER COMMENTARY

Experimental Design:

In addition to answering the questions posed, a focus of this experiment is to take multiple readings on each explanatory variable in order to illustrate the assumptions necessary for inference for regression. For this reason it is necessary for each group to make many drops from each initial height. We suggest at least 20 drops from each height. We also suggest at least four different initial heights so that a pattern of linearity may be established.

This experiment may be done when first studying linear regression or experimental design. In this case, the students would complete the Descriptive Statistics section first. The data could be revisited later when studying inference for regression.

Class discussion prior to developing a design should focus on such problems as:

- Is it appropriate to do all drops from one height before moving to the next height or should all 80 trials be randomized? What are the benefits and drawbacks of each method?
- Should the same person drop the ball each time? Should the same ball be used?
- How should measurement be accomplished? Is it sufficient for one person to measure all drops, or should there be more than one person measuring on each drop? Should the same people measure each drop?
- Should practice drops be done before starting the experiment?
- Consider the scope of inference. What variables are most important to consider?
- What other sources of variation are present in this experiment? Which are most important to control for?

Descriptive Statistics:

Ideally the scatterplot and residuals will indicate a linear model is appropriate. If not, a transformation of the data or re-examination of the data collection process may be necessary.

The slope indicates the increase in rebound height per unit increase in initial height.

The y-intercept would be the mean rebound height when the initial height is zero. Of course, when the initial height is zero, the ball is on the ground and it cannot rebound. However, the intercept may be significantly different from zero. If this is the case, students can clearly see the problem of drawing inference outside the range of the data.

Inferential Statistics:
The assumptions for inference for regression are:

- The values of the *y*-variable for each value of the *x*-variable are normally distributed.
- The mean values of the *y*-variables are linearly related.
- The values of the *y*-variables associated with a given *x*-variable are independent of one another.
- The variance of the *y*-variables is the same for each *x*-variable.

The distribution of the *y*-variables at each *x*-variable should be normally distributed and the variances of these distributions should be approximately equal. Because of the robustness of the t-procedures, some departure from these assumptions is acceptable. Students should address this issue, whether or not the assumptions are fully met.

The summary statistics for regression should include estimates and standard errors for both the slope and the *y*-intercept.

The hypothesis test should include null and alternative hypotheses and a calculation of the t-statistic:

$$t = \frac{b}{se_b}$$

Conclusions should be stated in the context of the problem. Include a statement relating the *p*-value of the hypothesis test to the confidence interval. Scope of inference should be discussed here.

If each group has a different type of ball, the regression equations should be compared. See if students can match equations with balls.